OCR Further Pure 1

Complex Numbers

Section 3: Modulus and argument

Solutions to Exercise

1. (i) $z = -2\sqrt{3} - 2i$

$|z|^2 = (2\sqrt{3})^2 + 2^2 = 12 + 4 = 16$

$|z| = 4$

$\arctan\left(\frac{2}{2\sqrt{3}}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

z is in the 3rd quadrant so $\arg z = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$

(ii) $z = 1 - 3i$

$|z|^2 = 1^2 + 3^2 = 1 + 9 = 10$

$|z| = \sqrt{10}$

$\arctan\left(\frac{-3}{1}\right) = -1.25$ (3 s.f.)

z is in the 4th quadrant so $\arg z = -1.25$ (3 s.f.)

(iii) $z = -3 + 3i$

$|z|^2 = 3^2 + 3^2 = 9 + 9 = 18$

$|z| = 3\sqrt{2}$

$\arctan\left(\frac{1}{-1}\right) = -\frac{\pi}{4}$

z is in the 2nd quadrant so $\arg z = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

2. (i) $r = 3, \quad \theta = \frac{\pi}{4}$

$x = r\cos\theta = 3\cos\frac{\pi}{4} = \frac{3}{\sqrt{2}}$

$y = r\sin\theta = 3\sin\frac{\pi}{4} = \frac{3}{\sqrt{2}}$

$z = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i$

(ii) $r = 6, \quad \theta = \frac{2\pi}{3}$

$x = r\cos\theta = 6\cos\left(\frac{2\pi}{3}\right) = 6 \times -\frac{1}{2} = -3$

$y = r\sin\theta = 6\sin\left(\frac{2\pi}{3}\right) = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$

$z = -3 + 3\sqrt{3}i$

(iii) $r = 2, \quad \theta = -\frac{\pi}{6}$

$x = r\cos\theta = 2\cos\left(-\frac{\pi}{6}\right) = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

$y = r\sin\theta = 2\sin\left(-\frac{\pi}{6}\right) = 2 \times -\frac{1}{2} = -1$

$z = \sqrt{3} - i$

3. (i) $z = 1 + 2i$

$|z| = \sqrt{1^2 + 2^2} = \sqrt{5}$

z is in the first quadrant, so $\arg z = \arctan\left(\frac{2}{1}\right) = 1.11$ (3 s.f.)

(ii) $z^* = 1 - 2i$

$|z^*| = \sqrt{1^2 + 2^2} = \sqrt{5}$

z^* is in the fourth quadrant, so $\arg z^* = \arctan\left(\frac{-2}{1}\right) = -1.11$ (3 s.f.)

(iii) $\frac{1}{z} = \frac{1}{1+2i} = \frac{1-2i}{(1+2i)(1-2i)} = \frac{1-2i}{5}$

$\left|\frac{1}{z}\right| = \frac{1}{5}\sqrt{1^2 + 2^2} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$

$\frac{1}{z}$ is in the fourth quadrant, so $\arg\frac{1}{z} = \arctan\left(\frac{-2}{1}\right) = -1.11$ (3 s.f.)

(iv) $\frac{1}{z^*} = \frac{1}{1-2i} = \frac{1+2i}{(1-2i)(1+2i)} = \frac{1+2i}{5}$

$\left|\frac{1}{z^*}\right| = \frac{1}{5}\sqrt{1^2 + 2^2} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$

$\frac{1}{z^*}$ is in the first quadrant, so $\arg\frac{1}{z^*} = \arctan\left(\frac{2}{1}\right) = 1.11$ (3 s.f.)

$$\left|\frac{1}{z}\right| = \left|\frac{1}{z^*}\right| = \frac{1}{|z|} = \frac{1}{|z^*|}$$

and $\arg z = \arg\frac{1}{z^*} = -\arg z^* = -\arg\frac{1}{z}$

4. $w = 10i, \quad z = 1+\sqrt{3}i$

$|w| = 10$

$\arg w = \frac{\pi}{2}$

$|z| = \sqrt{1+3} = 2$

z is in the first quadrant so $\arg z = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$wz = 10i(1+\sqrt{3}i) = 10i - 10\sqrt{3} = -10\sqrt{3} + 10i$

$|wz| = 10\sqrt{3+1} = 20$

wz is in the second quadrant so $\arg(wz) = \arctan\left(\frac{1}{-\sqrt{3}}\right) + \pi = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$

$$\frac{w}{z} = \frac{10i}{1+\sqrt{3}i} = \frac{10i(1-\sqrt{3}i)}{(1+\sqrt{3}i)(1-\sqrt{3}i)} = \frac{10i+10\sqrt{3}}{1+3} = \frac{5\sqrt{3}+5i}{2}$$

$\left|\frac{w}{z}\right| = \frac{5}{2}\sqrt{3+1} = 5$

$\frac{w}{z}$ is in the first quadrant so $\arg\left(\frac{w}{z}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

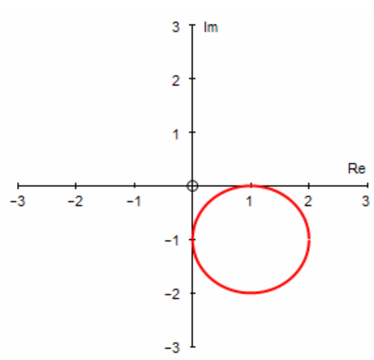
$|wz| = |w||z|, \quad \left|\frac{w}{z}\right| = \frac{|w|}{|z|}$

$\arg(wz) = \arg w + \arg z, \quad \arg\left(\frac{w}{z}\right) = \arg w - \arg z$

5. (i) $|z-1+i| = 1$

$|z-(1-i)| = 1$

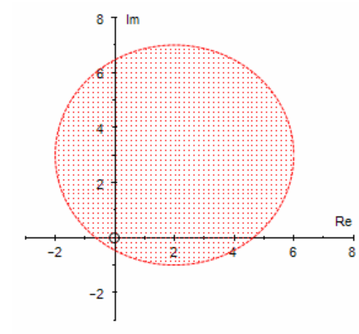
This is a circle, centre 1 – i, radius 1.



(ii) $|z-2-3i|<4$

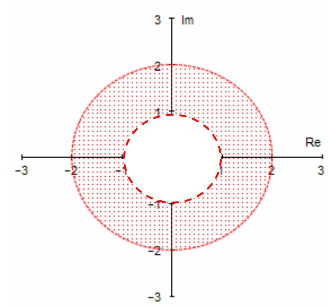
$|z-(2+3i)|<4$

This is the interior of a circle, centre $2+3i$, radius 4.



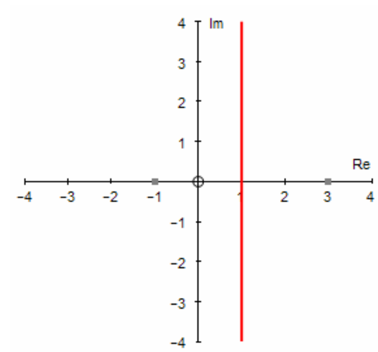
(iii) $1<|z|<2$

This is the region in between two circles, centres the origin, radii 1 and 2 respectively.



(iv) $|z+1|=|z-3|$

This is the perpendicular bisector of a line joining the points (3, 0) and (-1, 0).



6. $|z|=|z-2| \Rightarrow x^2+y^2=(x-2)^2+y^2$

$\Rightarrow x^2=x^2-4x+4$

$\Rightarrow 4x=4$

$\Rightarrow x=1$

$|z-i|=|z-1| \Rightarrow x^2+(y-1)^2=(x-1)^2+y^2$

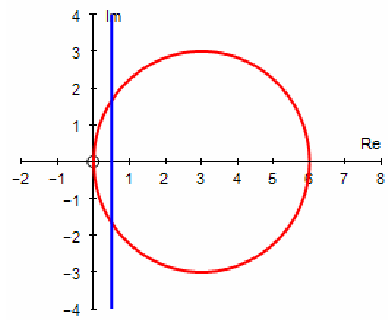
$\Rightarrow x^2+y^2-2y+1=x^2-2x+1+y^2$

$\Rightarrow -2y=-2x$

$\Rightarrow y=x$

The point of intersection of the lines $x=1$ and $y=x$ is (1, 1) so the value of z that satisfies both equations is $1+i$.

7. $|z-3|=3$ is a circle, centre 3, radius 3.
$|z|=|z-2|$ is the perpendicular bisector of the line joining the origin and the point (2, 0).



The equation of the circle is $(x-3)^2+y^2=9$
The equation of the line is $x=1$.

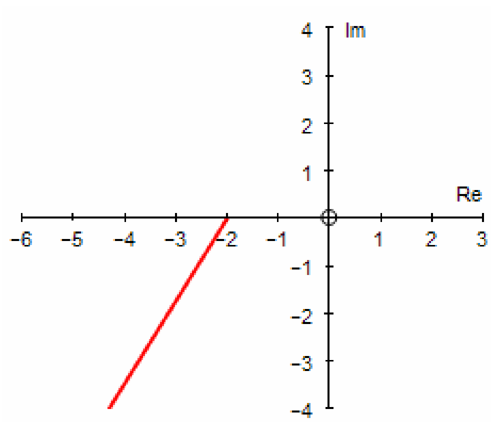
At the points of intersection,

$$(1-3)^2+y^2=9$$
$$4+y^2=9$$
$$y=\pm\sqrt{5}$$

The complex numbers satisfying both equations are $1+\sqrt{5}i$ and $1-\sqrt{5}i$.

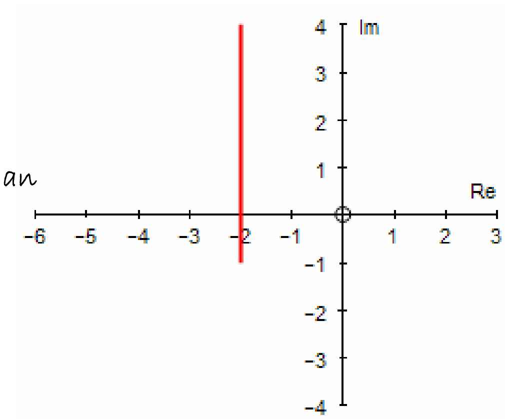
8. (i) $\arg(z+2)=-\dfrac{2\pi}{3}$

This is a half-line, starting at -2, at an angle of $\dfrac{2\pi}{3}$ below the positive real axis.



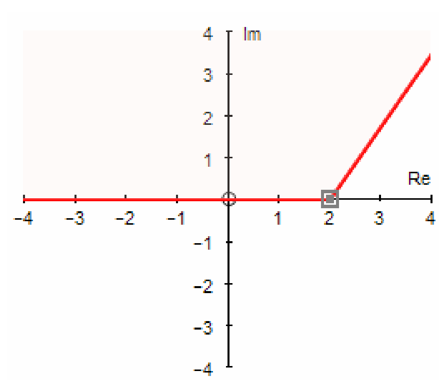
(ii) $\arg(z+2+i)=\dfrac{\pi}{2}$

This is a half-line, starting at -2 – i, at an angle of $\dfrac{\pi}{2}$ above the positive real axis.



(iii) $\dfrac{\pi}{3}\le\arg(z-2)\le\pi$

The boundaries of this region are two half-lines, both starting at 2, one at an angle to $\dfrac{\pi}{3}$ to the positive real axis, and the other on the negative real axis.



9. $|z+2+i| = |z-4+i|$

$(x+2)^2 + (y+1)^2 = (x-4)^2 + (y+1)^2$

$x^2 + 4x + 4 = x^2 - 8x + 16$

$12x = 12$

$x = 1$

$y = x\tan\theta = x\tan\frac{\pi}{4} = x$

When $x = 1$, $y = 1$
so the complex number is $1 + i$.