OCR Further Pure 1

Complex Numbers

Section 1: Introduction to complex numbers

Solutions to Exercise

1. (i) $z^2 + 25 = 0$

$z^2 = -25$

$z = \pm 5i$

(ii) $4z^2 + 9 = 0$

$z^2 = -\frac{9}{4}$

$z = \pm \frac{3}{2} i$

(iii) $z^2 - 2z + 2 = 0$

$$z = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

(iv) $4z^2 + 4z + 5 = 0$

$$z = \frac{-4 \pm \sqrt{16 - 4 \times 4 \times 5}}{8} = \frac{-4 \pm \sqrt{-64}}{8} = \frac{-4 \pm 8i}{8} = -\tfrac{1}{2} \pm i$$

2. (i) (a) $z_1 + z_2 = 2 + 3i + 1 - 2i = 3 + i$

(b) $z_1 - z_2 = 2 + 3i - 1 + 2i = 1 + 5i$

(c) $z_1 z_2 = (2 + 3i)(1 - 2i) = 2 - 4i + 3i + 6 = 8 - i$

(d) $z_1^* = 2 - 3i$

(e) $z_2^* = 1 + 2i$

(f) $z_1^* + z_2^* = 2 - 3i + 1 + 2i = 3 - i$

(g) $z_1^* - z_2^* = 2 - 3i - 1 - 2i = 1 - 5i$

(h) $z_1 * z_2 * = (2-3i)(1+2i) = 2+4i-3i+6 = 8+i$

(ii) (a) $z_1 + z_2 = -2i+3+i = 3-i$

(b) $z_1 - z_2 = -2i-3-i = -3-3i$

(c) $z_1 z_2 = -2i(3+i) = -6i+2 = 2-6i$

(d) $z_1 * = 2i$

(e) $z_2 * = 3-i$

(f) $z_1 * + z_2 * = 2i+3-i = 3+i$

(g) $z_1 * - z_2 * = 2i-(3-i) = -3+3i$

(h) $z_1 * z_2 * = 2i(3-i) = 6i+2 = 2+6i$

$z_1 * + z_2 * = (z_1 + z_2)*$

$z_1 * - z_2 * = (z_1 - z_2)*$

$z_1 * z_2 * = (z_1 z_2)*$

3. $z = (a+i)^4$

$= a^4 + 4a^3 i + 6a^2 i^2 + 4ai^3 + i^4$

$= a^4 + 4a^3 i - 6a^2 - 4ai + 1$

If z is real, $4a^3 - 4a = 0$

$4a(a^2 - 1) = 0$

$4a(a+1)(a-1) = 0$

so a = 0, -1 or 1.

4. (i) $\frac{1}{3+2i} + \frac{1}{3-2i} = \frac{3-2i+3+2i}{(3+2i)(3-2i)}$

$= \frac{6}{9+4}$

$= \frac{6}{13}$

(ii) $3+i+\frac{4}{3-i} = 3+i+\frac{4(3+i)}{(3-i)(3+i)}$

$= 3+i+\frac{4(3+i)}{9+1}$

$= 3+i+\frac{2}{5}(3+i)$

$= \frac{7}{5}(3+i)$

(iii) $$\frac{3}{1-i} - \frac{2i}{2+i} = \frac{3(1+i)}{(1-i)(1+i)} - \frac{2i(2-i)}{(2+i)(2-i)}$$
$$= \frac{3+3i}{2} - \frac{4i+2}{5}$$
$$= \frac{15+15i-8i-4}{10}$$
$$= \frac{11+7i}{10}$$

Note: if you have covered chapter 8 on the roots of polynomial equations you can do this question by considering the sum and product of the roots.

5. (i) One root is $2 - i$ so the other root is $2 + i$

Equation is $(z-2+i)(z-2-i)=0$
$$(z-2)^2+1=0$$
$$z^2-4z+4+1=0$$
$$z^2-4z+5=0$$
$p = -4$, $q = 5$

(ii) One root is $1 - 3i$ so the other root is $1 + 3i$

Equation is $(z-1+3i)(z-1-3i)=0$
$$(z-1)^2+9=0$$
$$z^2-2z+1+9=0$$
$$z^2-2z+10=0$$
$p = -2$, $q = 10$

(iii) One root is $2i$ so the other root is $-2i$

Equation is $(z-2i)(z+2i)=0$
$$z^2+4=0$$
$p = 0$, $q = 4$

(iv) One root is $5 - 3i$ so the other root is $5 + 3i$

Equation is $(z-5+3i)(z-5-3i)=0$
$$(z-5)^2+9=0$$
$$z^2-10z+25+9=0$$
$$z^2-10z+34=0$$
$p = -10$, $q = 34$

6. $\dfrac{5}{a+bi}+\dfrac{2}{1+3i}=1$

$\dfrac{5}{a+bi}=1-\dfrac{2}{1+3i}=\dfrac{1+3i-2}{1+3i}=\dfrac{-1+3i}{1+3i}$

$\dfrac{a+bi}{5}=\dfrac{1+3i}{-1+3i}$

$=\dfrac{(1+3i)(-1-3i)}{(-1+3i)(-1-3i)}$

$=\dfrac{-1-6i+9}{1+9}$

$=\dfrac{8-6i}{10}$

$=\dfrac{4-3i}{5}$

$a+bi=4-3i$

7. $(a+bi)^*=(a+bi)^2$

$a-bi=a^2+2abi-b^2$

Equating imaginary parts: $-b=2ab$

$b+2ab=0$

$b(1+2a)=0$

$b=0$ or $a=-\frac{1}{2}$

Equating real parts: $a=a^2-b^2$

If $b=0$: $a=a^2$

$a(1-a)=0$

$a=0$ or 1

If $a=-\frac{1}{2}$: $-\frac{1}{2}=\frac{1}{4}-b^2$

$b^2=\frac{3}{4}$

$b=\pm\frac{1}{2}\sqrt{3}$

The possible values of a and b are: $a=b=0$

$a=1,\ b=0$

$a=-\frac{1}{2},\ b=\pm\frac{1}{2}\sqrt{3}$

8. (i) $(a+bi)(2+i)=a-3i$

$2a+ai+2bi-b=a-3i$

Equating real parts: $2a-b=a$

$a=b$

Equating imaginary parts: $a+2b=-3$

$3a=-3$

$a=-1$

$a = -1$, $b = -1$

(ii) $(a+i)(4-bi) = 3b + 2ai$

$4a - abi + 4i + b = 3b + 2ai$

Equating real parts: $4a + b = 3b$

$2a = b$

$-ab + 4 = 2a$

$-2a^2 + 4 = 2a$

Equating imaginary parts: $a^2 + a - 2 = 0$

$(a+2)(a-1) = 0$

$a = -2$ or $a = 1$

$a = -2$, $b = -4$ or $a = 1$, $b = 2$

9. $(1+i)z - iw = 6$ ①

$iz + (1-i)w = 6 + 3i$ ②

① × $(1-i)$ $(1-i)(1+i)z - i(1-i)w = 6(1-i)$

② × i $-z + i(1-i)w = 6i - 3$

Adding: $(1-i)(1+i)z - z = 6 - 6i + 6i - 3$

$2z - z = 3$

$z = 3$

Substituting into ①: $3(1+i) - iw = 6$

$3 + 3i - iw = 6$

$iw = -3 + 3i$

Multiplying by i: $-w = -3i - 3$

$w = 3 + 3i$