OCR Further Pure 1

Complex Numbers

Solutions to Topic assessment

1. $(2+i)z+(3-2i)z^* = 32$

   Let $z = x + iy$

   $(2+i)(x+iy)+(3-2i)(x-iy) = 32$

   $2x+ix+2iy-y+3x-2ix-3iy-2y = 32$

   $5x-3y-ix-iy = 32$

   Equating imaginary parts: $-x-y=0 \Rightarrow y=-x$

   Equating real parts: $5x-3y=32 \Rightarrow 5x+3x=32$

   $\Rightarrow 8x = 32$

   $\Rightarrow x=4, \quad y=-4$

   so $z = 4 - 4i$.

2. $\alpha = -2+5i$

   (i) $\alpha^* = -2-5i$

   (ii) $|\alpha| = \sqrt{2^2+5^2} = \sqrt{29}$

   α is in the second quadrant, so $\arg\alpha = \pi + \arctan\left(\frac{5}{-2}\right)$

   $= \pi - 1.19$

   $= 1.95$ (3 s.f.)

   (iii) $$\frac{\alpha+\alpha^*}{\alpha} = \frac{-2+5i-2-5i}{-2+5i} = \frac{-4}{-2+5i} = \frac{-4(-2-5i)}{(-2+5i)(-2-5i)} = \frac{8+20i}{29} = \frac{8}{29}+\frac{20}{29}i$$

3. (i) $z_1^2 - 4z + 5 = (2+i)^2 - 4(2+i) + 5$

   $= 4+4i-1-8-4i+5$

   $= 0$

   The other root, $z_2 = z_1^* = 2-i$

(ii) $\frac{1}{z_1}+\frac{1}{z_2}=\frac{1}{2+i}+\frac{1}{2-i}$

$=\frac{2-i+2+i}{(2+i)(2-i)}$

$=\frac{4}{4+1}$

$=\frac{4}{5}$

(iii) $z_1^2+z_2^2=(2+i)^2+(2-i)^2$

$=4+4i-1+4-4i-1$

$=6$

$\text{Im}\left(z_1^2+z_2^2\right)=0$

$z_1^2-z_2^2=(2+i)^2-(2-i)^2$

$=4+4i-1-4+4i+1$

$=8i$

$\text{Re}\left(z_1^2-z_2^2\right)=0$

4. (i) $f(z)=z^3+z^2+4z-48$

$f(3)=3^3+3^2+4\times3-48$

$=27+9+12-48$

$=0$

3 is a root, so the real root $\alpha=3$.

$z^3+z^2+4z-48=0$

$(z-3)(z^2+4z+16)=0$

The complex roots are the roots of the quadratic equation

$z^2+4z+16=0$.

$z=\frac{-4\pm\sqrt{4^2-4\times1\times16}}{2}$

$=\frac{-4\pm\sqrt{-48}}{2}$

$=\frac{-4\pm4i\sqrt{3}}{2}$

$=-2\pm2i\sqrt{3}$

The complex roots are $\beta=-2+2i\sqrt{3}$, $\gamma=-2-2i\sqrt{3}$.

(ii) $|\alpha|=3$

$\arg\alpha=0$

$|\beta| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4$

β is in the second quadrant, so $\arg\beta = \pi + \arctan\left(\frac{2\sqrt{3}}{-2}\right)$

$= \pi + \arctan\left(-\sqrt{3}\right)$

$= \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

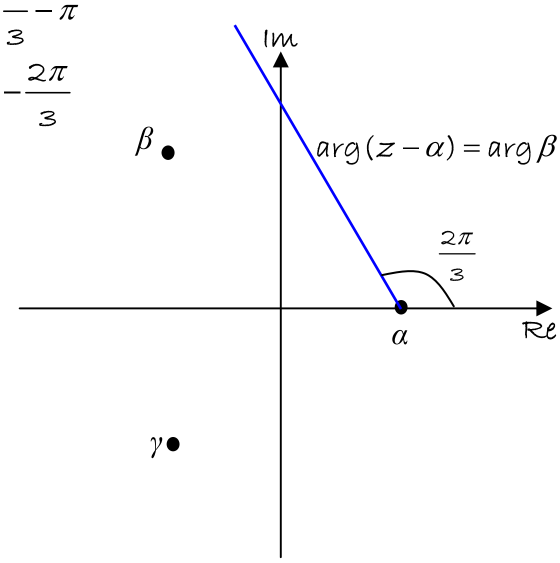
$|\gamma| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4$

γ is in the third quadrant, so $\arg\gamma = \arctan\left(\frac{-2\sqrt{3}}{-2}\right) - \pi$

$= \arctan\left(\sqrt{3}\right) - \pi$

$= \frac{\pi}{3} - \pi$

$= -\frac{2\pi}{3}$



(iii) $\arg(z-\alpha) = \arg\beta$

$\arg(z-3) = \frac{2\pi}{3}$

The locus is the half-line starting from $z = 3$ in the direction $\frac{2\pi}{3}$ (shown on Argand diagram above).

5. (i) $\alpha = 1 + 4i$

$\alpha^2 = (1+4i)(1+4i) = 1 + 8i - 16 = -15 + 8i$

$\alpha^3 = (-15+8i)(1+4i) = -15 - 52i - 32 = -47 - 52i$

(ii) $\alpha^3 + 5\alpha^2 + k\alpha + m = 0$

$-47 - 52i + 5(-15+8i) + k(1+4i) + m = 0$

Equating imaginary parts: $-52 + 40 + 4k = 0$

$\Rightarrow k = 3$

Equating real parts: $-47-75+k+m=0$

$\Rightarrow m=122-k=119$

(iii) $\alpha=1+4i$ is a root, so $\alpha^*=1-4i$ is another root.

A quadratic factor is $(z-1-4i)(z-1+4i)=(z-1)^2+16$

$$=z^2-2z+1+16$$

$$=z^2-2z+17$$

$z^3+5z^2+3z+119=0$

$(z^2-2z+17)(z+7)=0$

The third root is $z = -7$.

$\arg\alpha=\arctan\left(\frac{4}{1}\right)=1.33$ (3 s.f.)

By symmetry $\arg\alpha^*=-1.33$ (3 s.f.)

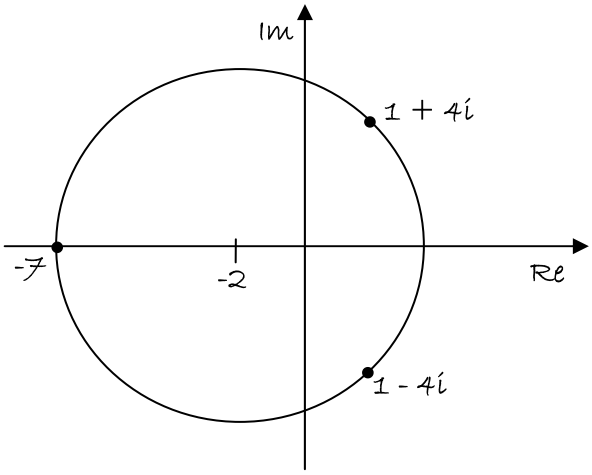
$\arg(-7)=\pi$.

(iv) $|\alpha+2|=|1+4i+2|=|3+4i|=\sqrt{3^2+4^2}=5$

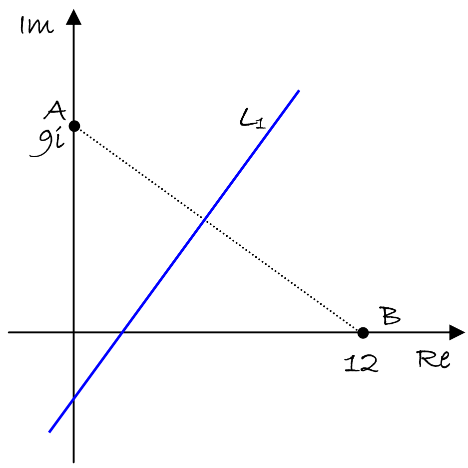
$|\alpha^*+2|=|1-4i+2|=|3-4i|=\sqrt{3^2+4^2}=5$

$|-7+2|=|-5|=5$

so all three roots satisfy $|z+2|=5$.



6. (i) L_1 is the perpendicular bisector of a line joining the points z = 12 and z = 9i on the Argand diagram.



(ii) $|z - 9i| = 2|z - 12|$

$$|x + iy - 9i| = 2|x + iy - 12|$$

$$\sqrt{x^2 + (y-9)^2} = 2\sqrt{(x-12)^2 + y^2}$$

$$x^2 + (y-9)^2 = 4\left((x-12)^2 + y^2\right)$$

$$x^2 + y^2 - 18y + 81 = 4x^2 - 96x + 576 + 4y^2$$

$$3x^2 - 96x + 3y^2 + 18y + 495 = 0$$

$$x^2 - 32x + y^2 + 6y + 165 = 0$$

$$(x-16)^2 - 256 + (y+3)^2 - 9 + 165 = 0$$

$$(x-16)^2 + (y+3)^2 = 100$$

This is a circle, centre (16, -3), radius 10.

(iii) $|z - 16 + 3i| = 10$