

EdExcel Further Pure 1

Complex Numbers

Section 3: Complex numbers and equations

Solutions to Exercise

1. $3 + i$ is a root, so $3 - i$ is also a root.

$$\begin{aligned}\text{Therefore a quadratic factor is } (z - 3 - i)(z - 3 + i) &= (z - 3)^2 + 1 \\ &= z^2 - 6z + 10\end{aligned}$$

- $1 + 3i$ is a root, so $1 - 3i$ is also a root.

$$\begin{aligned}\text{Therefore a quadratic factor is } (z - 1 - 3i)(z - 1 + 3i) &= (z - 1)^2 + 9 \\ &= z^2 - 2z + 10\end{aligned}$$

So the equation is $(z^2 - 6z + 10)(z^2 - 2z + 10) = 0$

$$z^4 - 8z^3 + 32z^2 - 80z + 100 = 0$$

2. $1 - 2i$ is a root, so $1 + 2i$ is also a root.

$$\begin{aligned}\text{So a quadratic factor is } (z - 1 + 2i)(z - 1 - 2i) &= (z - 1)^2 + 4 \\ &= z^2 - 2z + 5\end{aligned}$$

$$z^3 + z + 10 = (z^2 - 2z + 5)(z + 2)$$

The real root is $z = -2$.

3. $(1 + i)^2 = 1 + 2i - 1 = 2i$

$$(1 + i)^3 = 2i(1 + i) = 2i - 2 = -2 + 2i$$

Substituting into $z^3 - 2z + k = 0$:

$$-2 + 2i - 2(1 + i) + k = 0$$

$$-2 + 2i - 2 - 2i + k = 0$$

$$k = 4$$

$1 + i$ is a root, so $1 - i$ is also a root.

$$\begin{aligned}\text{So a quadratic factor is } (z - 1 - i)(z - 1 + i) &= (z - 1)^2 + 1 \\ &= z^2 - 2z + 2\end{aligned}$$

$$z^3 - 2z + 4 = (z^2 - 2z + 2)(z + 2)$$

so the other two roots are $1 - i$ and -2 .

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4. $z = -1 + i$

$$z^2 = (-1 + i)^2 = 1 - 2i - 1 = -2i$$

$$z^3 = -2i(-1 + i) = 2i + 2 = 2 + 2i$$

$$z^4 = (2 + 2i)(-1 + i) = -2 - 2 = -4$$

Substituting into $z^4 - 2z^3 - z^2 + 2z + 10$:

$$\begin{aligned} & -4 - 2(2 + 2i) - (-2i) + 2(-1 + i) + 10 \\ & = -4 - 4 - 4i + 2i - 2 + 2i + 10 \\ & = 0 \end{aligned}$$

so $-1 + i$ is a root.

Since $-1 + i$ is a root, $-1 - i$ is also a root

So a quadratic factor is $(z + 1 - i)(z + 1 + i) = (z + 1)^2 + 1$

$$= z^2 + 2z + 2$$

$$z^4 - 2z^3 - z^2 + 2z + 10 = (z^2 + 2z + 2)(z^2 - 4z + 5)$$

The other factors are the roots of the quadratic equation $z^2 - 4z + 5 = 0$

$$\begin{aligned} z &= \frac{4 \pm \sqrt{16 - 4 \times 1 \times 5}}{2} \\ &= \frac{4 \pm \sqrt{-4}}{2} \\ &= \frac{4 \pm 2i}{2} \\ &= 2 \pm i \end{aligned}$$

So the other roots are $-1 - i$, $2 + i$ and $2 - i$.

5. $z = p + qi$

$$z^2 = (p + qi)^2 = p^2 + 2pqi - q^2 = p^2 - q^2 + 2pqi$$

$$z^3 = (p^2 - q^2 + 2pqi)(p + qi)$$

$$= p(p^2 - q^2) + (2p^2q + q(p^2 - q^2))i - 2pq^2$$

$$= p^3 - 3pq^2 + (3p^2q - q^3)i$$

Substituting into $z^3 + az + b = 0$:

$$p^3 - 3pq^2 + (3p^2q - q^3)i + a(p + qi) + b = 0$$

(i) Equating imaginary parts: $3p^2q - q^3 + aq = 0$

$$3p^2 - q^2 = -a$$

(ii) Equating real parts: $p^3 - 3pq^2 + ap + b = 0$

$$p^3 - 3pq^2 + p(-3p^2 + q^2) + b = 0$$

$$-2p^3 - 2pq^2 = -b$$

$$2p(p^2 + q^2) = b$$

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(iii) From (i), $q^2 = 3p^2 + a$

Substituting into the result from (ii):

$$2p(p^2 + 3p^2 + a) = b$$

$$2p(4p^2 + a) = b$$

$$8p^3 + 2ap - b = 0$$

so p is a root of the equation $8x^3 + 2ax - b = 0$

6. $(a + bi)^2 = 3 - 4i$

$$a^2 + 2abi - b^2 = 3 - 4i$$

Equating imaginary parts: $2ab = -4 \Rightarrow a = -\frac{2}{b}$

Equating real parts: $a^2 - b^2 = 3$

$$\frac{4}{b^2} - b^2 = 3$$

$$4 - b^4 = 3b^2$$

$$b^4 - 3b^2 - 4 = 0$$

$$(b^2 - 4)(b^2 + 1) = 0$$

$$b = \pm 2$$

$$b = \pm 2 \Rightarrow a = \mp 1$$

The square roots of $3 - 4i$ are $1 - 2i$ and $-1 + 2i$.

7. $(a + bi)^* = (a + bi)^2$

$$a - bi = a^2 + 2abi - b^2$$

Equating imaginary parts: $-b = 2ab$

$$b + 2ab = 0$$

$$b(1 + 2a) = 0$$

$$b = 0 \text{ or } a = -\frac{1}{2}$$

Equating real parts: $a = a^2 - b^2$

If $b = 0$: $a = a^2$

$$a(1 - a) = 0$$

$$a = 0 \text{ or } 1$$

If $a = -\frac{1}{2}$: $-\frac{1}{2} = \frac{1}{4} - b^2$

$$b^2 = \frac{3}{4}$$

$$b = \pm \frac{1}{2}\sqrt{3}$$

The possible values of a and b are: $a = b = 0$

$$a = 1, b = 0$$

$$a = -\frac{1}{2}, b = \pm \frac{1}{2}\sqrt{3}$$

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8. (i) $(a + bi)(2 + i) = a - 3i$

$$2a + ai + 2bi - b = a - 3i$$

Equating real parts: $2a - b = a$

$$a = b$$

Equating imaginary parts: $a + 2b = -3$

$$3a = -3$$

$$a = -1$$

$$a = -1, b = -1$$

(ii) $(a + i)(4 - bi) = 3b + 2ai$

$$4a - abi + 4i + b = 3b + 2ai$$

Equating real parts: $4a + b = 3b$

$$2a = b$$

$$-ab + 4 = 2a$$

$$-2a^2 + 4 = 2a$$

Equating imaginary parts: $a^2 + a - 2 = 0$

$$(a + 2)(a - 1) = 0$$

$$a = -2 \text{ or } a = 1$$

$$a = -2, b = -4 \text{ or } a = 1, b = 2$$