# EdExcel Further Pure 1

## Complex Numbers

## Section 2: The Argand diagram and the modulus-argument form

### Notes and Examples

These notes contain subsections on:



### Representing complex numbers geometrically

The Notes and Examples for Section 1 looked at the relationships between numbers as represented in a Venn diagram, with some sets of numbers being a subset of another set: e.g. the integers are a subset of the rational numbers. You have seen that all the types of number that you have met so far can be considered to be a subset of a larger set of numbers: the complex numbers. This can be represented on the Venn diagram by a larger set encircling the set representing the real numbers.

Another way to represent numbers is on a number line. You have probably used number lines from a very early stage in your mathematical development. Even irrational numbers can be placed on a number line: for example, $\sqrt{2}$ can be expressed to as many decimal places as you like.

However, if you want to place a complex number on the number line, you have a problem. Is 1 + i larger or smaller than 1? Clearly this kind of question just does not make sense.

The Argand diagram provides a way of representing complex numbers geometrically, in the same way that a number line can represent the real numbers.

**Example 1**
The complex numbers $z$ and $w$ are given by

$z = 3 - 2\text{i}$

$w = -1 + 4\text{i}$

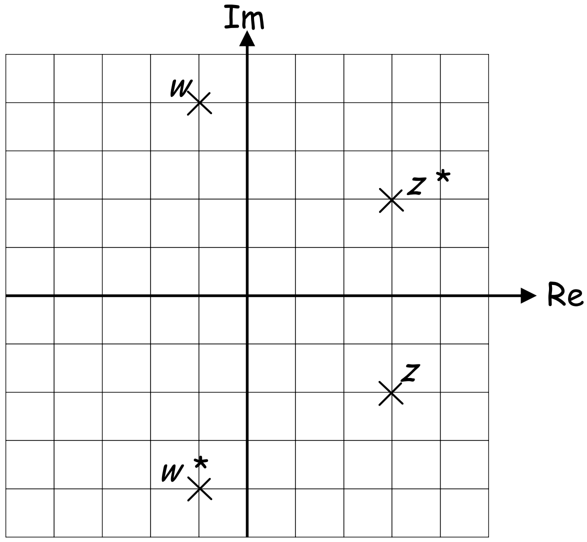
Plot the points $z$, $w$, $z^*$ and $w^*$ on an Argand diagram.

**Solution**
$z = 3 - 2i$ is represented by the point (3, -2)

$z^* = 3 + 2i$ is represented by the point (3, 2)

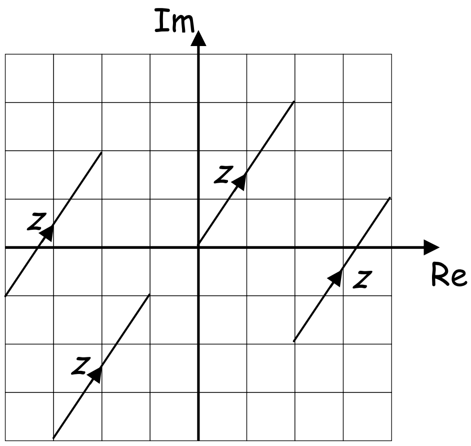
$w = -1 + 4i$ is represented by the point (-1, 4)

$w^* = -1 - 4i$ is represented by the point (-1, -4)




You can explore complex numbers in the Argand diagram using the Flash resource **The Argand diagram**.

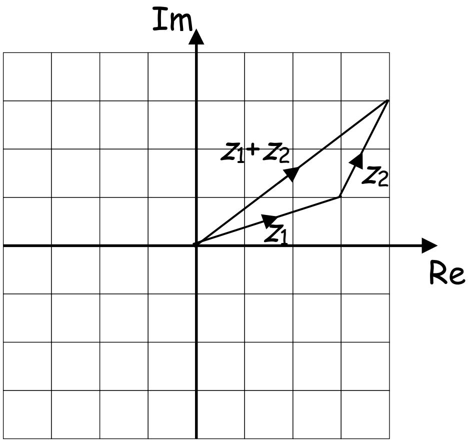
As well as thinking of a complex number $z = x + yi$ as a point with coordinates $(x, y)$, you can also think of it as a vector $\begin{pmatrix} x \\ y \end{pmatrix}$. This could be a position vector (a vector from the origin to the point $(x, y)$) but it can be any vector (sometimes called a directed line segment) parallel to this.




All the vectors on this diagram represent the complex number $z = 2 + 3i$. Notice that it is the vector itself that is labelled $z$, not the point at the end of it.

Don't worry if you don't know very much about vectors (they are covered in some depth in Core 4) – all you need to know here is how to represent the addition and subtraction of vectors on a diagram.

### Addition of two complex numbers

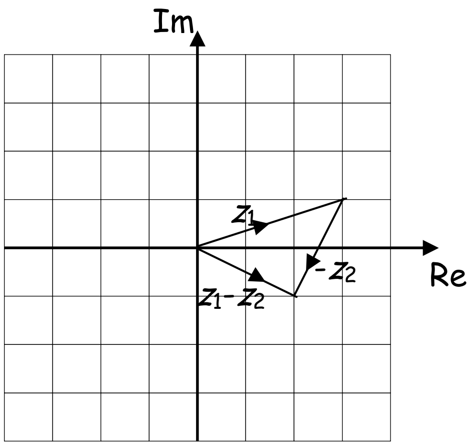


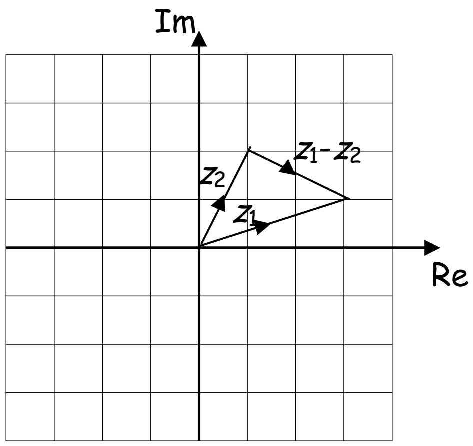

Here $z_1 = 3 + \mathrm{i}$ and $z_2 = 1 + 2\mathrm{i}$. You can see from the diagram that $z_1 + z_2 = 4 + 3\mathrm{i}$, as you would expect from adding $z_1$ and $z_2$ together.

### Subtraction of two complex numbers

You can think of subtraction in two different ways: either by thinking of $z_1 - z_2$ as adding together the vectors $z_1$ and $-z_2$ (shown in the diagram on the left) or by going from the point $z_2$ to the point $z_1$ (shown in the diagram on the right).

In either case, with $z_1 = 3 + \mathrm{i}$ and $z_2 = 1 + 2\mathrm{i}$, you can see that the vector $z_1 - z_2$ is given by $2 - \mathrm{i}$, the result you would expect from subtracting the complex number $z_2$ from $z_1$.




## The modulus of a complex number

You are familiar with describing a point in the plane using Cartesian coordinates. However, this is not the only way of describing the location of a point. One alternative is to give its distance from a fixed point (usually the origin) and a direction (in this case the angle between the line connecting the point to the origin, and the positive real axis).

This is a common method of describing locations in real life: you might say that a town is “50 miles north-west of London”, or when walking in open countryside your map might show you that you need to walk 2 miles on a bearing of 124°.

In mathematics there are some situations in which this method of describing points is more convenient than Cartesian coordinates.

The modulus of a complex number $z$ is the distance of the point representing $z$ from the origin on the Argand diagram. Notice that this definition also holds for real numbers on the number line: the modulus (or absolute value) of a real number is its distance on the number line from zero.

In the same way, $|z - w|$ (or $|w - z|$) is the distance of the point representing $z$ from the point representing $w$. This also holds for real numbers on the number line: the distance of a real number $x$ from a real number $y$ on the number line is $|x - y|$ (or $|y - x|$). For example, the distance between $2$ and $-3$ on the number line is $|2 - (-3)| = 5$.

When you multiply two complex numbers, then the modulus of the product is the product of the moduli of the two complex numbers.

$$|wz| = |w||z|$$

Similarly when you divide two complex numbers:

$$\left|\frac{w}{z}\right| = \frac{|w|}{|z|}$$

**Example 2**
Given that $z = 2 + 5\text{i}$ and $w = 3 - \text{i}$, find
(i) $|z|$
(ii) $|w|$
(iii) $|z - w|$
(iv) $|zw|$

**Solution**
(i) $|z| = \sqrt{2^2 + 5^2} = \sqrt{29}$

(ii) $|w| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$

(iii) $z - w = 2 + 5\text{i} - (3 - \text{i}) = -1 + 6\text{i}$
$|z - w| = \sqrt{(-1)^2 + 6^2} = \sqrt{37}$

(iv) $|zw| = |z||w| = \sqrt{29} \times \sqrt{10} = \sqrt{290}$

## The argument of a complex number

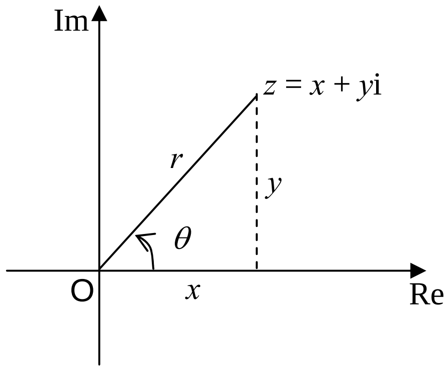
Finding the argument of a complex number involves using some knowledge of Trigonometry from C2, including radians, and angles greater than 90°. You also need to know the values of the sine, cosine and tangent for common angles such as 30° $\left(\frac{\pi}{6}\text{ radians}\right)$, 45° $\left(\frac{\pi}{4}\text{ radians}\right)$ and 60° $\left(\frac{\pi}{3}\text{ radians}\right)$.
However, if you haven't covered this work yet, don't worry. Click **here** for the Trigonometry notes which give some help on these topics.

The diagram shows that the argument $\theta$ of the complex number $z = x + y\text{i}$ satisfies the equations

$$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$$

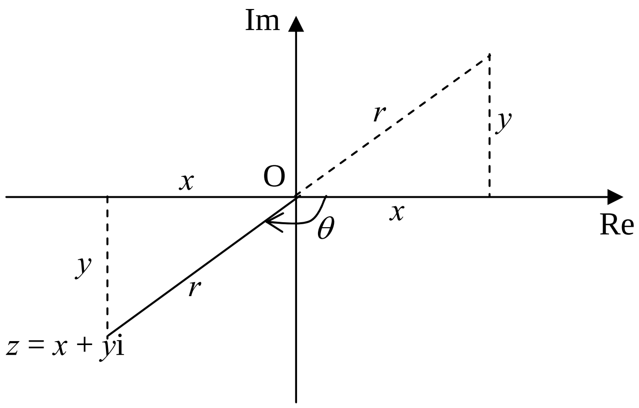
$$\sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\tan\theta = \frac{y}{x}$$



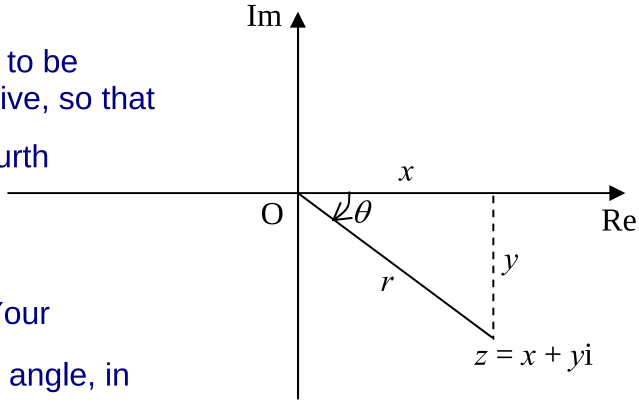

In the diagram, the complex number $z$ lies in the first quadrant, since both $x$ and $y$ are positive. So tan $\theta$ is positive, and to find the value of $\theta$, you just need to find $\arctan\frac{y}{x}$.

However, there is another possibility for which tan $\theta$ is positive. If both the real part and the imaginary part of $z$ are negative, $-\pi < \theta \le -\frac{\pi}{2}$. In this case $z$ is in the third quadrant.
However, as tan $\theta$ is positive, using your calculator to find $\arctan\frac{y}{x}$ will give you the corresponding angle in the first quadrant, (see diagram) where $x$ and $y$ are both positive. To find the correct argument, you need to subtract $\pi$.




Next we need to look at the cases where tan $\theta$ is negative.

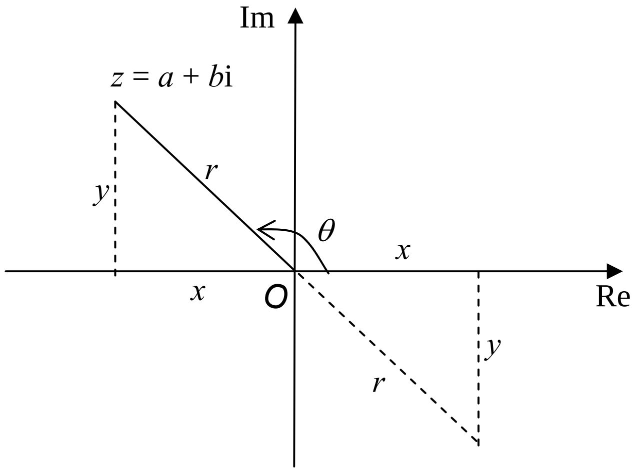
One possibility is for the real part of $z$ to be positive and the imaginary part negative, so that $-\frac{\pi}{2} \leq \theta < 0$. In this case, $z$ is in the fourth quadrant.




To find the value of $\theta$, find $\arctan \frac{y}{x}$. Your calculator should give you a negative angle, in the fourth quadrant as required.

The value of $\tan \theta$ is also negative if the real part of $z$ is negative, but the imaginary part is positive, so that $\frac{\pi}{2} \leq \theta \leq \pi$. In this case $z$ is in the second quadrant.

Using your calculator to find $\arctan \frac{y}{x}$ will give you the corresponding angle in the fourth quadrant, with $x$ positive and $y$ negative (see diagram). To find the correct angle in the second quadrant, you need to add $\pi$.




**Example 3**

Find the modulus and argument of each of the following complex numbers.

(i) $z = 4 + 3\text{i}$ (ii) $z = -1 + \text{i}$ (iii) $z = -1 - \sqrt{3}\text{i}$

**Solution**

(i) $z = 4 + 3\text{i}$

$|z| = \sqrt{(3^2 + 4^2)} = 5$

Since $z$ lies in the first quadrant, arg $z = \arctan \frac{3}{4} \approx 0.644$

(ii) $z = -1 + \text{i}$

$|z| = \sqrt{(1^2 + 1^2)} = \sqrt{2}$

Since $z$ lies in the second quadrant, arg $z = \arctan(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

(iii) $z = -1 - \sqrt{3}\text{i}$

Modulus = $\sqrt{1+3} = 2$

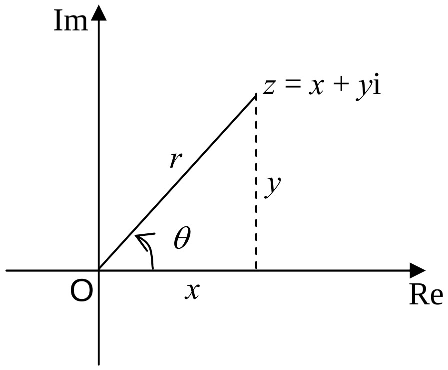
Since $z$ lies in the fourth quadrant, $\arg z = \arctan\sqrt{3} - \pi = \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$

Sometimes you may want to find a complex number if you are given its modulus and argument.

You can see from the diagram that

$x = r\cos\theta$

$y = r\sin\theta$




These relationships allow you to find the real and imaginary parts of a complex number with a given modulus and argument.

**Example 4**

Find the complex numbers with the given modulus and argument, in the form $x + \mathrm{i}y$.

(i) $|z| = 3, \quad \arg z = \frac{3\pi}{4}$

(ii) $|z| = 2, \quad \arg z = -\frac{\pi}{6}$

**Solution**

(i) $r = 3,\ \theta = \frac{3\pi}{4}$

$$x = 3\cos\frac{3\pi}{4} = 3 \times -\frac{1}{\sqrt{2}} = -\frac{3}{\sqrt{2}}$$

$$y = 3\sin\frac{3\pi}{4} = 3 \times \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$z = -\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}\mathrm{i}$$

(ii) $r = 2,\ \theta = -\frac{\pi}{6}$

$$x = 2\cos\left(-\frac{\pi}{6}\right) = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$y = 2\sin\left(-\frac{\pi}{6}\right) = 2 \times -\frac{1}{2} = -1$$

$$z = \sqrt{3} - \mathrm{i}$$

# EdExcel FP1 Complex nos Sec. 2 Notes & Examples

You can test yourself on this work using the interactive questions **Complex number: polar form**.

You can also try the **Complex numbers puzzle**.


## Multiplying and dividing using the modulus-argument form (extension work)

You have already met the relationships

$$|wz| = |w||z| \quad \text{and} \quad \left|\frac{w}{z}\right| = \frac{|w|}{|z|}$$

which you need to know for your examination.

It is also possible to find the argument of the product or quotient of complex numbers, using the relationships

$$\arg wz = \arg w + \arg z$$

$$\arg \frac{w}{z} = \arg w - \arg z$$

You will not be tested on the relationships for the arguments in your examination.

These results allow you to multiply and divide complex numbers in the modulus-argument form, quickly and easily.

These results can be interpreted geometrically using the Argand diagram:

- When $z$ is multiplied by $w$, the vector $z$ is enlarged by a scale factor $|w|$ and rotated through an angle of $\arg w$ anticlockwise about O
- When $z$ is divided by $w$, the vector $z$ is enlarged by a scale factor $\frac{1}{|w|}$ and rotated through an angle of $\arg w$ clockwise about O.

These ideas are demonstrated in the Flash resource **Multiplying and dividing in the Argand diagram**.

Example 5 shows how these relationships can be used.

**Example 5**

For the complex numbers $z = 3\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)$ and $w = 4\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)$, find, in polar form,

(i) $wz$ (ii) $\frac{w}{z}$.

Illustrate the points $w$, $z$, $wz$ and $\frac{w}{z}$ on an Argand diagram.

**Solution**

$|z| = 3, |w| = 4$

$\arg z = \frac{2\pi}{3}, \arg w = \frac{\pi}{2}$

(i) $|wz| = |w| \times |z| = 3 \times 4 = 12$

$\arg wz = \arg w + \arg z$

$= \frac{\pi}{2} + \frac{2\pi}{3}$

$= \frac{7\pi}{6}$

This cannot be the principal argument of $zw$ as it is not in the range $-\pi < \theta \le \pi$. The value must be adjusted by adding or subtracting multiples of $2\pi$.

Principal argument of $wz = \frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$

$$wz = 12\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$$

(ii) $\left|\frac{w}{z}\right| = \frac{|w|}{|z|} = \frac{4}{3}$

$\arg\frac{w}{z} = \arg w - \arg z$

$= \frac{\pi}{2} - \frac{2\pi}{3}$

$= -\frac{\pi}{6}$

This time the argument is in the correct range so does not need adjusting.

$$\frac{w}{z} = \frac{4}{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$