

EdExcel Further Pure 1

Complex Numbers

Section 1: Introduction to complex numbers

Notes and Examples

These notes contain subsections on

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The number system

In your learning of mathematics, you have come across different types of number at different stages. Each time you were introduced to a new set of numbers, this allowed you to solve a wider range of problems.

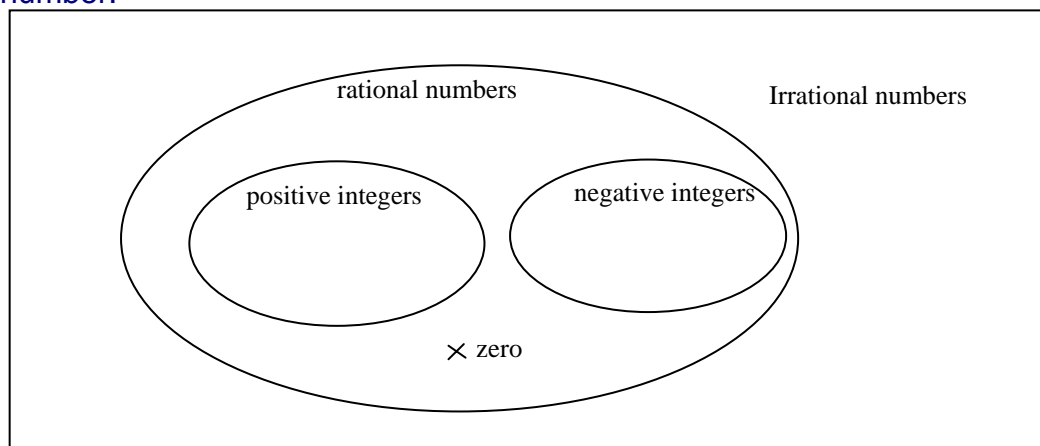
The first numbers that you came across were the counting numbers (natural numbers). These allowed you to solve equations like $x + 2 = 5$.

Later you would meet negative numbers, which allowed you to solve equations like $x + 5 = 2$, and rational numbers, which meant you could solve equations like $2x = 5$.

When irrational numbers were included, you could solve equations like $x^2 = 2$.

However, there are still equations which you cannot solve, such as $x^2 = -4$. You know that there are no real numbers which satisfy this equation. However, this equation, and others like it, can be solved using imaginary numbers, which are based on the number i , which is defined as $\sqrt{-1}$.

The diagram below shows the relationships between different types of number.



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This type of diagram is called a Venn diagram (you may have met these before if you have studied any Statistics) and it shows the relationships between sets, in this case sets of numbers. This diagram deals with the real numbers, which include all numbers which you have come across until now. Notice that the positive and negative integers (whole numbers) are subsets of the rational numbers. This means that all integers are also rational numbers, but there are other rational numbers which are not integers, such as $\frac{3}{2}$ or $-\frac{7}{11}$. Similarly, all rational numbers are real numbers, but there are other real numbers which are not rational, such as $\sqrt{3}$ and π .

In this chapter you will see that the real numbers are also a subset of a larger set called the complex numbers. You will be looking at numbers which lie outside the set of real numbers. Complex numbers involve both real and imaginary numbers.

Adding and subtracting complex numbers

To add two complex numbers, you need to add the real parts and add the imaginary parts. Similarly, to subtract one complex number from another, deal with the real and imaginary parts separately.



Example 1

The complex numbers z and w are given by

$$z = 3 + 2i$$

$$w = 1 - 4i$$

Find:

(i) $z + w$

(ii) $z - w$

(iii) $w - z$



Solution

$$\begin{aligned} \text{(i) } z + w &= (3 + 2i) + (1 - 4i) \\ &= (3 + 1) + (2i - 4i) \\ &= 4 - 2i \end{aligned}$$

Add the real parts and
add the imaginary parts

$$\begin{aligned} \text{(ii) } z - w &= (3 + 2i) - (1 - 4i) \\ &= (3 - 1) + (2i + 4i) \\ &= 2 + 6i \end{aligned}$$

Subtract the real parts and
subtract the imaginary parts

$$\begin{aligned} \text{(iii) } w - z &= (1 - 4i) - (3 + 2i) \\ &= (1 - 3) + (-4i - 2i) \\ &= -2 - 6i \end{aligned}$$



For practice in examples like the one above, try the interactive resource [Addition and subtraction of complex numbers.](#)

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Quadratic equations with complex roots

When you first learned to solve quadratic equations using the quadratic formula, you found that some quadratic equation had no real solutions. However, using complex numbers you can find solve all quadratic equations.



Example 2

Solve the quadratic equation

$$x^2 + 6x + 13 = 0$$

Solution

Using the quadratic formula with $a = 1$, $b = 6$, $c = 13$

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\&= \frac{-6 \pm \sqrt{36 - 4 \times 1 \times 13}}{2 \times 1} \\&= \frac{-6 \pm \sqrt{-16}}{2} \\&= \frac{-6 \pm 4i}{2} \\&= -3 \pm 2i\end{aligned}$$

The solutions of the equation are $x = -3 + 2i$ and $x = -3 - 2i$



Look at the Flash resource [Complex roots of a quadratic](#) which shows how the complex roots of a quadratic equation can be related to the graph of the quadratic function. You do not need to know this work, but it is interesting extension work.

Multiplying complex numbers

Multiplication of two complex numbers is similar to multiplying out a pair of brackets. Each term in the first bracket must be multiplied by each term in the second bracket. You can then simplify, remembering that $i^2 = -1$.



Example 3

Find

- (i) $(3 + 4i)(2 + i)$
- (ii) $(2 - i)(3 + 2i)$
- (iii) $(2 + 3i)(2 - 3i)$

Solution

$$\begin{aligned}\text{(i)} \quad (3 + 4i)(2 + i) &= 6 + 3i + 8i + 4i^2 \\&= 6 + 11i - 4 \\&= 2 + 11i\end{aligned}$$

Multiply out the brackets

Using $i^2 = -1$



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$$\begin{aligned} \text{(ii)} \quad (4 - i)(3 + 2i) &= 12 + 8i - 3i - 2i^2 \\ &= 12 + 5i + 2 \\ &= 14 + 5i \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad (2 + 3i)(2 - 3i) &= 4 - 6i + 6i - 9i^2 \\ &= 4 + 9 \\ &= 13 \end{aligned}$$



For practice in examples like the one above, try the interactive resource [Multiplication of complex numbers](#).

Complex conjugates

In part (iii) of Example 3, the result of multiplying two complex numbers is a real number. This is always the case when the complex number $a + bi$ is multiplied by the complex number $a - bi$. The complex number $a - bi$ is called the **complex conjugate** of $a + bi$. For any complex number z , the complex conjugate is written as \bar{z} or z^* .

Notice that if a quadratic equation has complex roots, then these are a pair of complex conjugates. All quadratic equations with real coefficients have two solutions: either two real solutions (which could be a repeated solution) or two complex solutions which are a pair of complex conjugates.



For practice in complex conjugates, try the interactive resource [Conjugate of a complex number](#).

The next example shows how you can find a quadratic equation with roots at particular complex values. A quadratic equation with roots at $x = a$ and $x = b$ can be written as $(x - a)(x - b) = 0$, and this also applies to situations where the roots are complex numbers.

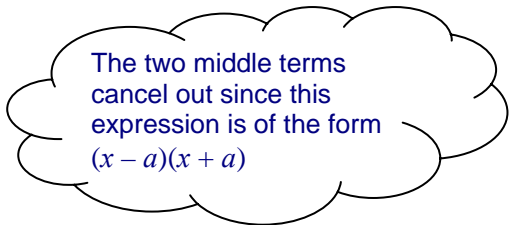


Example 4

Find the quadratic equation which has roots at $x = 4 + 2i$ and $x = 4 - 2i$.

Solution

$$\begin{aligned} (x - (4 + 2i))(x - (4 - 2i)) &= 0 \\ (x - 4 - 2i)(x - 4 + 2i) &= 0 \\ (x - 4)^2 - (2i)^2 &= 0 \\ x^2 - 8x + 16 + 4 &= 0 \\ x^2 - 8x + 20 &= 0 \end{aligned}$$



The two middle terms cancel out since this expression is of the form $(x - a)(x + a)$

Division of complex numbers

The result that a complex number multiplied by its conjugate is always real is key to the technique of dividing complex numbers. By multiplying both the

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numerator and the denominator by the complex conjugate of the denominator, the denominator becomes real.



Example 5

Write $\frac{2+3i}{3-4i}$ in the form $a+bi$.

Solution

$$\begin{aligned}\frac{2+3i}{3-4i} &= \frac{(2+3i)(3+4i)}{(3-4i)(3+4i)} \\ &= \frac{6+8i+9i-12}{9+16} \\ &= \frac{-6+17i}{25} \\ &= -\frac{6}{25} + \frac{17}{25}i\end{aligned}$$

Multiply top and bottom by $3+4i$ (the complex conjugate of $3-4i$)



For practice in examples like the one above, try the interactive resource [Dividing complex numbers](#).

The next example shows how you can solve an equation involving complex numbers using the four operations.



Example 6

Solve the equation

$$(3-2i)(z-1+4i) = 7+4i$$

Solution

$$\begin{aligned}(3-2i)(z-1+4i) &= 7+4i \\ z-1+4i &= \frac{7+4i}{3-2i} = \frac{(7+4i)(3+2i)}{(3-2i)(3+2i)} \\ &= \frac{21+14i+12i-8}{9+4} \\ &= \frac{13+26i}{13} = 1+2i \\ z &= 1+2i - (-1+4i) \\ &= 2-2i\end{aligned}$$

Divide both sides by $3-2i$

Subtract $-1+4i$ from each side



The Flash resource [Working with complex numbers](#) tests you on multiplication, complex conjugates and equations with complex roots.