AQA Further Pure 1

Complex Numbers

Section 1: Introduction to Complex Numbers

Solutions to Exercise

1. (i) $z^2 + 25 = 0$

$z^2 = -25$

$z = \pm 5i$

(ii) $4z^2 + 9 = 0$

$z^2 = -\frac{9}{4}$

$z = \pm \frac{3}{2} i$

(iii) $z^2 - 2z + 2 = 0$

$$z = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2}$$
$$= \frac{2 \pm \sqrt{-4}}{2}$$
$$= \frac{2 \pm 2i}{2}$$
$$= 1 \pm i$$

(iv) $4z^2 + 4z + 5 = 0$

$$z = \frac{-4 \pm \sqrt{16 - 4 \times 4 \times 5}}{8}$$
$$= \frac{-4 \pm \sqrt{-64}}{8}$$
$$= \frac{-4 \pm 8i}{8}$$
$$= -\tfrac{1}{2} \pm i$$

2. (i) (a) $z_1 + z_2 = 2 + 3i + 1 - 2i = 3 + i$

(b) $z_1 - z_2 = 2 + 3i - 1 + 2i = 1 + 5i$

(c) $z_1 z_2 = (2 + 3i)(1 - 2i) = 2 - 4i + 3i + 6 = 8 - i$

(d) $z_1{}^* = 2 - 3i$

(e) $z_2{}^* = 1 + 2i$

(f) $z_1{}^* + z_2{}^* = 2 - 3i + 1 + 2i = 3 - i$

(g) $z_1{}^* - z_2{}^* = 2 - 3i - 1 - 2i = 1 - 5i$

(h) $z_1{}^* z_2{}^* = (2-3i)(1+2i) = 2+4i-3i+6 = 8+i$

(ii) (a) $z_1 + z_2 = -2i + 3 + i = 3 - i$

(b) $z_1 - z_2 = -2i - 3 - i = -3 - 3i$

(c) $z_1 z_2 = -2i(3+i) = -6i + 2 = 2 - 6i$

(d) $z_1{}^* = 2i$

(e) $z_2{}^* = 3 - i$

(f) $z_1{}^* + z_2{}^* = 2i + 3 - i = 3 + i$

(g) $z_1{}^* - z_2{}^* = 2i - (3-i) = -3 + 3i$

(h) $z_1{}^* z_2{}^* = 2i(3-i) = 6i + 2 = 2 + 6i$

$z_1{}^* + z_2{}^* = (z_1 + z_2)^*$
$z_1{}^* - z_2{}^* = (z_1 - z_2)^*$
$z_1{}^* z_2{}^* = (z_1 z_2)^*$

3. $z = (a+i)^4$

$= a^4 + 4a^3 i + 6a^2 i^2 + 4ai^3 + i^4$

$= a^4 + 4a^3 i - 6a^2 - 4ai + 1$

If z is real, $4a^3 - 4a = 0$

$4a(a^2 - 1) = 0$

$4a(a+1)(a-1) = 0$

so $a = 0, -1$ or 1.

4. $(a+bi)^* = (a+bi)^2$

$a - bi = a^2 + 2abi - b^2$

Equating imaginary parts: $-b = 2ab$

$b + 2ab = 0$

$b(1+2a) = 0$

$b = 0$ or $a = -\frac{1}{2}$

Equating real parts: $a = a^2 - b^2$

If $b = 0$: $a = a^2$

$a(1-a) = 0$

$a = 0$ or 1

If $a = -\frac{1}{2}$: $-\frac{1}{2} = \frac{1}{4} - b^2$

$b^2 = \frac{3}{4}$

$b = \pm\frac{1}{2}\sqrt{3}$

The possible values of a and b are: $a = b = 0$

$a = 1,\ b = 0$

$a = -\frac{1}{2},\ b = \pm\frac{1}{2}\sqrt{3}$

5. (i) $(a+bi)(2+i)=a-3i$

$2a+ai+2bi-b=a-3i$

Equating real parts: $2a-b=a$

$a=b$

Equating imaginary parts: $a+2b=-3$

$3a=-3$

$a=-1$

$a = -1,\ b = -1$

(ii) $(a+i)(4-bi)=3b+2ai$

$4a-abi+4i+b=3b+2ai$

Equating real parts: $4a+b=3b$

$2a=b$

$-ab+4=2a$

$-2a^2+4=2a$

Equating imaginary parts: $a^2+a-2=0$

$(a+2)(a-1)=0$

$a=-2$ or $a=1$

$a = -2,\ b = -4$ or $a = 1,\ b = 2$

6. $(a+bi)^2=3-4i$

$a^2+2abi-b^2=3-4i$

Equating imaginary parts: $2ab=-4 \quad \Rightarrow a=-\frac{2}{b}$

Equating real parts: $a^2-b^2=3$

$\frac{4}{b^2}-b^2=3$

$4-b^4=3b^2$

$b^4+3b^2-4=0$

$(b^2+4)(b^2-1)=0$

$b=\pm 1$

$b=\pm 1 \Rightarrow a=\mp 2$

The square roots of $3-4i$ are $2-i$ and $-2+i$.

7. The sum of the roots of the equation $z^2 + pz + q = 0$ is $-p$, and the product of the roots is q.

(i) One root is $2 - i$ so the other root is $2 + i$
Sum of roots is $2 - i + 2 + i = 4$ so $p = -4$
Product of roots is $(2 - i)(2 + i) = 4 + 1 = 5$ so $q = 5$

(ii) One root is $1 - 3i$ so the other root is $1 + 3i$
Sum of roots is $1 - 3i + 1 + 3i = 2$ so $p = -2$
Product of roots is $(1 - 3i)(1 + 3i) = 1 + 9 = 10$ so $q = 10$

(iii) One root is $2i$ so the other root is $-2i$
Sum of roots is 0 so $p = 0$
Product of roots is $2i \times -2i = 4$ so $q = 4$

(iv) One root is $5 - 3i$ so the other root is $5 + 3i$
Sum of roots is $5 - 3i + 5 + 3i = 10$ so $p = -10$
Product of roots is $(5 - 3i)(5 + 3i) = 25 + 9 = 34$ so $q = 34$