AQA Further Pure 1

Complex Numbers

Topic assessment

1. Find the complex roots of the equation $x^2 + 6x + 10 = 0$ [2]

2. Express in the form $a + \mathrm{i}b$:

   (a) $(2 + \mathrm{i})^2$; [1]

   (b) $(5 + 2\mathrm{i})(-3 + 4\mathrm{i})$. [1]

3. Given that $p, q \in \mathbb{R}$ find p and q where:

   $$2(p + \mathrm{i}q) = q - \mathrm{i}p - 2(1 - \mathrm{i})$$ [2]

4. (a) Show that $(3 - \mathrm{i})^2 = 8 - 6\mathrm{i}$. [1]

   (b) The quadratic equation

   $$az^2 + bz + 10\mathrm{i} = 0$$

   where a and b are real, has a root $3 - \mathrm{i}$.

   (i) Show that $a = 3$ and find the value of b. [6]

   (ii) Determine the other root of the quadratic equation, giving your answer in the form $p + \mathrm{i}q$. [3]

5. Given that $z = -2 + 2\sqrt{3}\mathrm{i}$, show that $z^2 + 4z$ is real. [3]

6. If $z = x + \mathrm{i}y$ and $z^* = x - \mathrm{i}y$, solve $(z + 5) - 2(z^* - 3) = 1 - 2\mathrm{i}$ [3]

7. If the roots of the equation $z^2 - (1 - \mathrm{i})z + 2 - \mathrm{i} = 0$ are α and β,

   (a) find and simplify the values of $\alpha + \beta$ and $\alpha\beta$. [2]

   (b) Find the equation whose roots are $\alpha + 2\beta$ and $2\alpha + \beta$. [6]

Total 30 marks

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Solutions to Topic assessment

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

1. In $x^2 + 6x + 10 = 0$

   $a = 1, b = 6$ and $c = 10$

$$x = \frac{-6 \pm \sqrt{36 - 4 \times 1 \times 10}}{2 \times 1}$$

$$x = \frac{-6 \pm 2i}{2} = -3 \pm i$$

The solutions are $x = -3 + i$ and $x = -3 - i$

Remember i × i = -1

2. (a) $(2+i)^2 = 4 + 4i + i^2 = 4 + 4i - 1 = 3 + 4i$

   (b) $(5+2i)(-3+4i) = -15 + 20i - 6i + 8i^2$

$$= -15 + 14i - 8$$

$$= -23 + 14i$$

3.

In general, when $z = p + q\mathrm{i}$, where p and q are real numbers,

- the real part of z is p, and
- The imaginary part of z is q.

$$2(p + iq) = q - ip - 2(1 - i)$$

$$2p + 2iq = q - ip - 2 + 2i$$

Equating real parts: $2p = q - 2$ ①

Equating imaginary parts: $2q = -p + 2$ or $p = 2 - 2q$ ②

Substituting ② into ①: $2(2 - 2q) = q - 2$

$$4 - 4q = q - 2$$

$$5q = 6$$

$$q = \tfrac{6}{5}$$

$$p = 2 - 2 \times \tfrac{6}{5} = -\tfrac{2}{5}$$

4. (a) $(3-i)^2 = 9-6i+i^2 = 9-6i-1 = 8-6i$

   (b) (i) Since $z = 3-i$ and $z^2 = 8-6i$,
   therefore $a(8-6i)+b(3-i)+10i = 0$

   Equating real parts: $8a + 3b = 0$ (1)
   Equating imaginary parts: $-6a - b + 10 = 0$ (2)

   Solving by (1) + 3×(2) gives $-10a + 30 = 0$
   $a = 3$ and $b = -8$

   (ii) Sum of roots $= -\dfrac{b}{a}$, in this case $\dfrac{8}{3}$
   therefore the other root must be $\dfrac{8}{3} - (3-i) = -\dfrac{1}{3} + i$.

5. $z = -2 + 2\sqrt{3}i$
   $4z = -8 + 8\sqrt{3}i$
   $z^2 = 4 - 8\sqrt{3}i - 12 = -8 - 8\sqrt{3}i$
   so $z^2 + 4z = -8 - 8\sqrt{3}i - 8 + 8\sqrt{3}i = -16$

6. $(z+5) - 2(z^* - 3) = 1 - 2i$
   $(x + iy + 5) - 2(x - iy - 3) = 1 - 2i$
   $-x + 11 + 3iy = 1 - 2i$

   Equating real parts: $-x + 11 = 1 \Rightarrow x = 10$

   Equating imaginary parts: $3y = -2 \Rightarrow y = -\dfrac{2}{3}$

   The solution is $z = 10 - \dfrac{2}{3}i$.

7. (a) In the quadratic equation $z^2 - (1-i)x + 2 - i = 0$:
   $a = 1, b = -(1-i)$ and $c = 2 - i$
   $\alpha + \beta = 1 - i$
   $\alpha\beta = 2 - i$

Remember: When the quadratic equation $ax^2 + bx + c = 0$ has roots α and β then the product of the roots, $\alpha\beta = \dfrac{c}{a}$ and the sum of the roots, $\alpha + \beta = -\dfrac{b}{a}$.

(b) For the new equation:

Sum of roots $= \alpha + 2\beta + 2\alpha + \beta$

$$= 3\alpha + 3\beta$$

$$= 3(\alpha + \beta)$$

$$= 3(1 - i)$$

Product of roots $= (\alpha + 2\beta)(2\alpha + \beta)$

$$= 2\alpha^2 + \alpha\beta + 4\alpha\chi + 2\beta^2$$

$$= 2(\alpha^2 + \beta^2) + 5\alpha\beta$$

$$= 2((\alpha + \beta)^2 - 2\alpha\beta) + 5\alpha\beta$$

$$= 2(\alpha + \beta)^2 + \alpha\beta$$

$$= 2(1 - i)^2 + 2 - i$$

$$= 2 - 4i - 2 + 2 - i$$

$$= 2 - 5i$$

For the new equation, putting $a = 1$ gives $b = -3(1 - i)$ and $c = 2 - 5i$ so the new equation is $z^2 - 3(1 - i)z + 2 - 5i = 0$