

Thursday's Answers

$$1. \quad 2007^{2007} \equiv 7^{2007} \pmod{100} \equiv \left((7^4)^{501} \times 7^3 \right) \pmod{100} \equiv \left((2401)^{501} \times 7^3 \right) \pmod{100} \\ \equiv (1 \times 343) \pmod{100} \equiv 43 \pmod{100}$$

Alternatively, look at pattern in last two digits of powers of 7:
07, 49, 43, 01, 07, ...

2. There are 7 such triangles

$$a^2 - b^2 = 2007^2 \Rightarrow (a-b)(a+b) = 3^4 \times 223^2$$

$$a-b=1, \quad a+b=3^4 \times 223^2 = 4028049 \Rightarrow a=2014025, b=2014024$$

$$a-b=3, \quad a+b=3^3 \times 223^2 = 1342683 \Rightarrow a=671343, b=671340$$

$$a-b=3^2, \quad a+b=3^2 \times 223^2 = 447561 \Rightarrow a=223785, b=223776$$

$$a-b=3^3, \quad a+b=3^1 \times 223^2 = 149187 \Rightarrow a=74157, b=74130$$

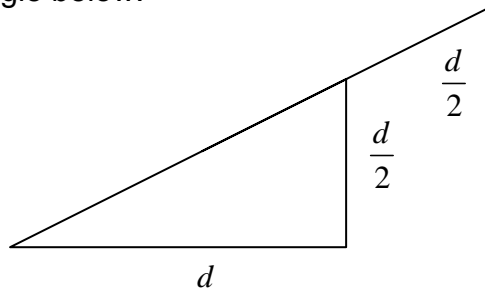
$$a-b=3^4, \quad a+b=223^2 = 49729 \Rightarrow a=24905, b=24824$$

$$a-b=223, \quad a+b=3^4 \times 223 = 18063 \Rightarrow a=9143, b=8920$$

$$a-b=3 \times 223, \quad a+b=3^3 \times 223 = 6021 \Rightarrow a=3345, b=2676$$

This completes the set since $a+b > a-b$

3. The radius marked R is perpendicular to the tangent of the two touching circles and so passes through the centres of both. This gives the right angled triangle below.



It follows that

$$R = \frac{d}{2} + \frac{d\sqrt{5}}{2} \Rightarrow \frac{R}{d} = \frac{1+\sqrt{5}}{2}$$

4. From $15n+1$ to $15(n+1)$ exactly 8 numbers will be said aloud.

$2007 = 8 \times 250 + 7$ so by the time the game reaches $15 \times 250 = 3750$ then $8 \times 250 = 2000$ numbers will have been said aloud. The seventh number to be said aloud after this will be $3750 + 13 = 3763$

5. The number of odd entries in row n of Pascal's Triangle is equal to $2^{b(n)}$ where $b(n)$ is the number of 1s in the binary expansion of the number n .

$$2007 = 1024 + 512 + 256 + 128 + 64 + 16 + 4 + 2 + 1$$

$$2007_{10} = 11111010111_2$$

Therefore this row contains $2^9 = 512$ odd entries and so $2008 - 512 = 1496$ even entries.

Friday's Answers

1. If $N=2007$ then 2007 is only three from the end with 2006, 2003 and 2002 being the final three numbers. So we need to continue until exactly three more numbers slot in before 2007. These will be 2008, 2009 and 2011. So N could be 2011, 2012 or 2013.

2. After 2008 stages it will be at $((2008/4)+1, 0) = (503, 0)$

Therefore, at stage 2007 it will be at $(0, -502)$

$$b\sqrt{2} = 3\sqrt{2} \times (1 + 2 + \dots + 502) = 3 \times 251 \times 503\sqrt{2} \Rightarrow b = 378759$$

Note that no other diagonal is an integer multiple of $\sqrt{2}$ since the sum of the squares of consecutive numbers is odd.

You may find <http://www.research.att.com/~njas/sequences/A001652> helpful in calculating a .

3. 2s and 5s pair off to produce multiples of 10 and since there are more 2s than 5s in the prime factorisation of 2007! we need to find the number of 5s in this factorisation.

2007/5 401 numbers are multiples of 5

2007/25 80 numbers are multiples of 5^2

2007/125 16 numbers are multiples of 5^3

2007/625 3 numbers are multiples of 5^4

$$401 + 80 + 16 + 3 = 500$$

4.

$$2007 = 3 \times 669 \quad \dots \quad 2007 = 668 + 669 + 670$$

$$2007 = 9 \times 223 \quad \dots \quad 2007 = 219 + 220 + 221 + 222 + 223 + 224 + 225 + 226 + 227$$

$$2007 = 223 \times 9 \quad \dots \quad 2007 = -102 + \dots + 0 + 1 + \dots + 120 = 103 + 104 + \dots + 120$$

$$2007 = 669 \times 3 \quad \dots \quad 2007 = -331 + \dots + 0 + 1 + \dots + 337 = 332 + 333 + 334 + 335 + 336 + 337$$

$$2007 = 2007 \times 1 \quad \dots \quad 2007 = -1002 + -1001 + \dots + 0 + 1 + \dots + 1004 = 1003 + 1004$$

So the longest run is $103 + 104 + \dots + 120$, i.e. 18 numbers.

5. Since the prime factorisation of 2007 is $3 \times 3 \times 223$, then $N = 2^{222} \times 3^2 \times 5^2$

Note these 2007 factors occur once each in the expansion of

$$(1 + 2 + 2^2 + \dots + 2^{222})(1 + 3 + 3^2)(1 + 5 + 5^2)$$

and so the sum of all the factors is $(2^{223} - 1) \times 13 \times 31 = 403 \times (2^{223} - 1) \approx 5.4 \times 10^{69}$